DISTRIBUTED MODEL REFERENCE ADAPTIVE CONTROL OF DISCRETE-TIME LINEAR TIME-INVARIANT SYSTEMS*

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Abstract. This paper proposes a distributed state feedback model reference adaptive control (MRAC) framework for discrete-time linear time-invariant uncertain multi-agent systems. Distinct from existing methods, this study pioneers the extension of conventional discrete-time MRAC methodology to effectively address distributed control scenarios. Taking the reference model as the leader to be tracked, each follower agent synthesizes a unified distributed MRAC law by utilizing state information available from its neighboring agents. Incorporating a gradient based parameter update law with partial knowledge of control gains, the proposed method ensures closed-loop stability and asymptotic leader-following tracking. Unlike conventional distributed observer based methods limited to autonomous leaders, this method allows external reference inputs for the leader. Particularly, the proposed method effectively manages parametric uncertainties in both leader and follower dynamics, addressing a notable limitation of existing methods that typically assume perfect knowledge of the leader's dynamics. Furthermore, to handle completely unknown control gains, a modified distributed MRAC law is developed by introducing an implicit estimate of the control gain parameter. This design leads to a linear regression form of the estimation error equation, thereby eliminating the need for prior knowledge of control gain signs or bounds in the gradient based parameter update law. This method avoids using Nussbaum gain functions, thus preventing potential adverse oscillations commonly associated with existing distributed adaptive protocols. Simulation results validate the theoretical findings.

Key words. Leader-following tracking, adaptive control, multi-agent systems, control gain uncertainty.

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1. Introduction.

In recent decades, the cooperative control of multi-agent systems (MASs) has emerged as a focal point within the control community due to its significance in some practical domains like smart grids, formation flying, and so on. In this direction, numerous remarkable works have been published to develop various cooperative control algorithms for MASs [1, 2, 3]. From the perspective of control objectives, consensus serves as the foundation for many sophisticated cooperative control tasks. To be specific, consensus control for MASs seeks to establish a uniform behavior throughout the agent network using distributed coordination mechanisms. This uniform behavior is determined either by all participants within the MASs or by a specified agent of the MASs, corresponding to leaderless consensus and leader-following tracking, respectively. To date, consensus control has attracted extensive investigation given its fundamental importance in the MASs research [4, 5, 6, 7, 8]. For example, [4] studied the graph conditions required for consensus convergence in

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MASs with linear dynamics. For second-order MASs under switching topology, [5] developed distributed observer method to address the leader-following tracking problem. Moreover, considering measurement noises, the works in [6, 7] addressed the leaderless consensus problem under binary-valued communication.

Initially, the design of consensus protocols relies on precise agent dynamics [4, 5, 6, 7, 8]. However, in practical applications, various system dynamic uncertainties cannot be ignored by designers [9, 10, 11, 12]. Consequently, using adaptive techniques to design consensus protocols of MASs for combating challenging system uncertainties has become a widely concerned research topic. Along this line, significant progress has been made over the past decades [13, 14, 15, 16, 17, 18]. For example, [15] designed a backstepping based adaptive distributed protocol for high-order nonlinear MASs subject to mismatched uncertainties. For MASs with autonomous leader dynamics, [16] studied the leader-following output synchronization problem by employing the distributed observer technique. To achieve angle-constrainted formation for uncertain nonholonomic mobile robots, [17] presents a user-friendly adaptive formation control scheme, reducing sensing links and computational cost largely. It is notable that some prior knowledge about the unknown control gain are necessary for adaptive consensus control when agent possesses uncertain control gain [15]. To overcome this, some Nussbaum gain based consensus algorithms have been proposed [3, 19, 20, 21, 22, 23, 24]. Specifically, [19, 20, 21] employ multiple Nussbaum functions under the assumption of identical control directions to achieve adaptive consensus without gain knowledge. For agents with possibly different control directions, compensator based networks in [22, 23] enable output regulation without prior gain information. Notably, these methods focus on continuous-time MASs, where Lyapunov based analysis is central to design adaptive parameter laws.

As is well known, model reference adaptive control (MRAC) is a widely studied technique for ensuring closed-loop stability and tracking of uncertain linear and nonlinear systems [25, 26, 27]. Recently, MRAC has been extended to MASs for consensus control [28, 29, 30, 31, 32, 33, 34, 35]. For example, [28] proposed an adaptive leaderless consensus scheme for linear MASs with matching uncertainties, while [29] developed a model reference based stabilizing consensus strategy for heterogeneous MASs. To achieve leader-following tracking, [30, 31, 32] designed distributed continuous-time MRAC schemes using state feedback. For instance, [31] designed a fictitious reference signal based state synchronization method for heterogeneous continuous-time linear MASs, where agent disconnected to the leader converged to a virtual model defined by its neighbors. Furthermore, [33] presented a direct distributed output feedback MRAC methodology for continuous-time MASs, relying on the communication of agents' control inputs among neighbors. Utilizing the indirect adaptive control technique, [34] introduced an indirect distributed MRAC framework for MASs in continuous-time dynamics, where system parameters of each agent are estimated for constructing virtual reference model and obtaining controller's parameter estimates. Notably, these distributed MRAC methods still require some prior knowledge of the control gain when designing adaptive laws, which is similar to the traditional MRAC [26, 27].

Despite considerable advancements in MASs consensus control, some open problems still exist. As is mentioned above, MRAC based distributed consensus designs have been extensively studied for continuous-time MASs. However, their discrete-time counterparts have been largely overlooked. Due to the essential stability differences between the continuous-time and discrete-time systems, the continuous-time MASs consensus methods cannot be directly transferred to the discrete-time counterparts.

Furthermore, existing distributed adaptive consensus algorithms face significant challenges: they typically require prior knowledge such as the signs and bounds of uncertain control gain parameters, or depend on oscillatory Nussbaum gain function methods for solving unknown sign information of the control gain parameters. These constraints significantly restrict their practical applicability. To address this gap, this study aims to propose a new distributed MRAC framework to achieve leader-following tracking control for a general class of uncertain discrete-time MASs. Taking the reference model as the leader agent, we will develop a unified distributed controller structure for all follower agents with uncertainties by using local neighboring information. The reference model is driven by an external control input thus being non-autonomous, which is typically different from common distributed observer based design methods [16, 36, 37]. Moreover, the dynamics of leader agent is also considered as uncertain with unknown parameters, which has not been studied in most existing literature related to distributed MRAC. Furthermore, we will consider that the control gain related parameter is completely unknown for each follower agent. In this case, based on our previous experience on addressing unknown control direction [39, 40], we will design a modified distributed MRAC law to achieve desired control performance without introducing Nussbaum gain function. In short, this paper's key innovative contributions are as follows.

(i) A new distributed MRAC framework is developed for a general class of uncertain discrete-time linear MASs. Based on neighbors' measurable signals, each follower agent constructs a parameterized virtual reference signal suitable for adaptive design. Then, with known control gain sign and bound knowledge, a distributed adaptive controller along with a gradient based parameter update law is developed to ensure closed-loop stability and achieve asymptotic leader-following output consensus.

(ii) Compared with existing MRAC based distributed consensus methods [30, 32, 33, 34, 35], the proposed strategy in this paper introduces a unified distributed controller structure for each follower in discrete-time MASs. Notably, the leader agent system considered in this study incorporates parameter uncertainties, whereas most existing methods assume exact knowledge of the leader's dynamics [32, 33, 34]. Furthermore, the proposed framework accommodates an external reference input for the leader agent, which distinguishes it from conventional distributed observer based methods that are typically applicable only to MASs with an autonomous leader [16, 36, 37].

(iii) To address the challenge posed by completely unknown control gains, a modified distributed MRAC law is proposed. The key idea is to incorporate the uncertain control gain into a new adaptive parameter, which leads to a linear regression structure for the estimation error equation. As a result, the design of parameter update law does not require prior knowledge of the control gain signs or bounds, which are typically required in most existing distributed consensus protocols [13, 15, 32, 33]. Moreover, unlike existing approaches that rely on Nussbaum gain functions to deal with unknown control directions, the proposed scheme avoids introducing Nussbaum dynamics and thus prevents the potential oscillatory behavior associated with such techniques [3, 19, 20, 21, 22, 23, 24].

The remaining contents of this study are as follows. Section 2 formulates the research problem, including the system model, the control objective, and the design conditions. Moreover, we outline the key technical issues to be solved. Section 3 presents the main results of the paper, where a nominal distributed controller and a unified distributed MRAC framework are first developed. Further, we design a modified distributed MRAC law to handle the challenge of completely unknown control gain parameter of follower agent. In this section, we detail the whole design process and present a systematic stability analysis. In Section 4, simulation study is conducted to validate the proposed methods. Finally, Section 5 concludes this paper.

Notation. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers. The symbols z and z^{-1} represent the time advance and time delay operators, respectively, i.e., $z[x](k) = x(k+1)$ and $z^{-1}[x](k) = x(k-1)$, where $k \in \{0, 1, 2, 3, ...\}$, $x(k) \triangleq x(kT)$ for a sampling period $T > 0$, and $x(k)$ is any finite dimensional signal. We define the signal spaces: $L^\infty = \{x(k) : \ ||x(\cdot)||_\infty < \infty, \ ||x(\cdot)||_\infty = \sup_{k \geq 0} \max_{1 \leq i \leq n} |x_i(k)|\}$ and $L^2 = \{x(k) : \ ||x(\cdot)||_2 < \infty, \ ||x(\cdot)||_2 = (\sum_{k=0}^{\infty} |x_1(k)|^2 + \cdots + |x_n(k)|^2)^{\frac{1}{2}}\}$ for $x(k) = [x_1(k), ..., x_n(k)]^T \in \mathbb{R}^n$. The symbol $c > 0$ denotes a generic signal bound, and $\tau(k)$ denotes a generic $L^2 \cap L^\infty$ function converging to zero as $k \to \infty$. For any vector signal $x(k)$, $\|x(k)\|$ denotes its Euclidean norm. The notation I denotes the identity matrix.

2. Problem statement. This section formulates the system model, the control objective, and assumptions, and articulates the key technical issues to be addressed.

2.1. System model. Consider a discrete-time MAS consisting of N follower agents and one leader agent. The dynamic model of the ith follower agent, $i = 1, ..., N$, is expressed as the following linear system

$$x_i(k+1) \ = \ A_i x_i(k) + b_i u_i(k), \ y_i(k) = c_i^T x_i(k), \tag{2.1}$$

where $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}$, and $y_i(k) \in \mathbb{R}$ are the state vector, control input, and system output, respectively, of agent i. Moreover, $A_i \in \mathbb{R}^{n_i \times n_i}$, $b_i \in R^{n_i}$, and $c_i \in \mathbb{R}^{n_i}$ denote system matrices with unknown parameters. From the state space model (2.1), we derive the inpuk-output description of agent i as

$$y_i(k) = G_i(z)[u_i](k), \ i = 1, ..., N, \tag{2.2}$$

where

$$G_i(z) = c_i^T (zI - A_i)^{-1} b_i = k_{p_i} \frac{Z_i(z)}{P_i(z)} \tag{2.3}$$

is the transfer function of the ith system dynamics. Particularly, $k_{p_i} \neq 0$ is called as the high-frequency gain of the system (2.2), i.e., the control gain parameter of the system (2.2), and $P_i(z)$ and $Z_i(z)$ are monic polynomials with unknown constant coefficients and of degrees n_i and m_i, respectively. Moreover, $n_i - m_i > 0$ is referred as the relative degree of agent i. In this paper, we consider that the state vector $x_i(k)$ is measurable and utilizable for every agent i, $i = 1, ..., N$.

The dynamics of the leader agent is described by a stable linear system model as

$$x_0(k+1) \ = \ A_0 x_0(k) + b_0 u_0(k), \ y_0(k) = c_0^T x_0(k), \tag{2.4}$$

where $x_0(k) \in \mathbb{R}^{n_0}$ is the state vector of the leader, $u_0(k) \in \mathbb{R}$ is a bounded external reference input signal, and $y_0(k) \in \mathbb{R}$ is the system output of the leader. Moreover, $A_0 \in \mathbb{R}^{n_0 \times n_0}$, $b_0 \in \mathbb{R}^{n_0}$ and $c_0 \in \mathbb{R}^{n_0}$ are time invariant matrices such that A_0 is stable and $G_0(z) = c_0^T (zI - A_0)^{-1} b_0 = k_{p0} \frac{Z_0(z)}{P_0(z)}$ has relative degree n^*. Particularly, this study assumes that A_0, b_0 and c_0 are unknown. Furthermore, it is assumed that the state variable $x_0(k)$ is available for the leader agent.

Remark 2.1. Unlike conventional MRAC frameworks that typically address tracking for a single plant [27], this study develops a distributed discrete-time MRAC for an MAS comprising N followers (2.1) and one leader (2.4). The leader serves as the reference, generating output via a bounded external input, and uncertainties are present in both follower (2.1) and leader (2.4) dynamics.

2.2. Graph theory. The communication network among $N+1$ agents (one leader and N followers) is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. Each vertex denotes an agent, and a directed edge (j, i) denotes that the agent i can receive the information from agent j (but not vice versa), making j a neighbor of i. Vertex 0 represents the leader agent. For each follower i, $i = 1, ..., N$, define its neighbor set $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{(N+1)\times(N+1)}$ satisfies $a_{ii} = 0$, and $a_{ij} = 1$ for $i \neq j$ if and only if $(j, i) \in \mathcal{E}$. In particular, for $i = 1, ..., N$, $a_{i0} = 1$ if and only if the agent i can access the information of leader agent 0. Moreover, d_i, $i, ..., N$, denotes the indegree of follower agent i, i.e., the number of agents in its neighbor set $\mathcal{N}_i$.

2.3. Control objective and assumptions.

Control objective. This paper aims to develop distributed adaptive control strategies based on local neighbor information for each follower agent, such that all signals in the closed-loop system are bounded and the outputs of all follower agents asymptotically track the output of leader, i.e., $\lim_{k\to\infty}(y_i(k) - y_0(k)) = 0$, $i = 1, ..., N$.

Assumptions. In order to achieve the control objective, the following design conditions are needed.

(A1) For each follower agent i, $i = 1, ..., N$, (A_i, b_i, c_i) is stabilizable and detetable.
(A2) The polynomial $Z_i(z)$ in (2.3) is stable for each follower agent i, $i = 1, ..., N$.
(A3) The degree n_i, $i = 0, ..., N$, is known and the relative degree of agent i is $n_i - m_i = n^*$ for $i = 1, ..., N$.
(A4) The directed graph $\mathcal{G}$ is acyclic and has at least one directed spanning tree with the leader agent being the root.

Assumption (A1) is to ensure output matching and internal signal boundedness [27]. Assumption (A2) is the so-called minimum phase condition essential for stable model reference control (MRC) in single-agent systems [27]. Assumption (A3) specifies the system degree n_i, which determines the degree of estimated controller parameters and enables the construction of a parameterized model. Moreover, it is necessary for the follower agents (2.1) to have identical system relative degree n^* equal to the leader (2.4) to ensure model matching. This condition, fundamental in conventional MRAC [27], is also adopted here for distributed MRAC design [30, 33, 34]. Finally, Assumption (A4) shows the interaction graph connectivity, which ensures well-definedness of the distributed control laws to be designed. This connectivity requirement is standard for some distributed control protocols [30, 33, 31]. Given Assumption (A4), the adjacency matrix A possesses an upper triangular form under a specific vertex permutation. For the convenience of following design, we order the agents such that the adjacency matrix A of graph $\mathcal{G}$ is upper triangular.

2.4. Technical issues. Now, we articulate the key technical issues in designing distributed MRAC framework for the MAS comprising of (2.1) and (2.4).

In MASs, some follower agents cannot access the leader's reference directly due to network connectivity. Therefore, each follower must generate a valid reference signal using only local neighbor information. To this end, distributed observers have been widely studied[16, 36], initially for cooperative output regulation without uncertainties and later extended to MASs with follower uncertainties [16, 36, 37]. For example, [16] proposed a distributed adaptive observer to estimate the leader's state and dynamics, enabling a virtual reference for adaptive control. However, these methods assume an autonomous leader; for leaders driven by external inputs $u_0(k)$, distributed observer may fail since the response of leader is jointly determined by the dynamics matrix and the reference input. We thus require a new reference signal design by utilizing local neighboring information to achieve tracking for the leader (2.4) with an external input $u_0(k)$.

In addition, the dynamic uncertainties of agents (2.1) and (2.4) in the MAS need to be addressed. Although some results have been made in leader-following adaptive tracking for continuous-time MASs [13, 15, 16, 30, 32, 31], there are few studies focusing on discrete-time MASs. This gap exists because most continuous-time methods rely on Lyapunov-based techniques to develop parameter adaptation laws and perform unified stability analysis of both tracking and estimation errors, which is not applicable to the discrete-time MAS described in (2.1) and (2.4). Moreover, while the existing distributed continuous-time MRAC methods can handle non-autonomous leader dynamics [30, 32, 33, 34, 31], some notable issues persist: The controller structures in [30, 32, 33, 34, 31] vary depending on whether the agents are leader-connected or leader-disconnected, and thus lack a unified MRAC architecture; Moreover, the methods in [32, 33, 34, 31] rely on the condition that the reference system model information is completely known. In contrast, this study considers uncertainties in both the follower agents (2.1) and the leader system model (2.4). Therefore, the development of a unified distributed MRAC framework capable of accommodating uncertainties across all agents represents a key challenge addressed in this work.

Furthermore, we investigate the situation in which the control gain parameter k_{p_i} in the system model (2.2) is completely unknown except for its non-zero nature, an aspect overlooked in some existing MASs consensus studies. Specifically, some prior works either neglect this gain uncertainty [13, 16, 17] or require its prior knowledge, i.e., the sign information of k_{p_i}, for stable adaptive control design [15, 30, 32, 33, 34, 35, 31]. Notably, the sign information of control gain parameter represents the control direction and may be unavailable in some practical engineering systems [43]. While some Nussbaum gain based MASs consensus methods addressed the difficulty of completely unknown control gain [3, 19, 20, 21, 22, 23, 24], such methods generally suffer from adverse oscillations due to inherent limitations of Nussbaum gain functions [38]. Therefore, it is imperative to design an adaptive leader-following tracking control method for the MAS (2.1) and (2.4) under the restriction of completely unknown control gain parameter k_{p_i} without relying on Nussbaum gain function. In this work, we devote to solve this issue by employing the method proposed in [39, 40], which was originally developed for single-agent systems.

With the aforementioned clarifications in mind, the following technical issues will be solved in this work.

- How to devise a virtual reference signal by using local neighboring information to achieve asymptotic tracking for the non-autonomous leader reference system (2.4) with parametric uncertainties;
- How to develop a unified distributed MRAC law for the discrete-time linear MAS (2.1) and (2.4) with the presence of parameter uncertainties affecting all agents, including both the follower agents and the leader system;
- How to address the uncertainty in control gain k_{p_i} when designing a unified distributed MRAC law for each follower agent that removes the need for prior

control gain parameters knowledge of the system (2.2) and does not rely on conventional Nussbaum gain functions.

3. Distributed MRAC designs. This section develops a distributed state feedback MRAC framework for the MAS consisting of (2.1) and (2.4). Considering that the state information can be shared among neighboring nodes through the interaction graph $\mathcal{G}$, we would first construct a nominal distributed MRC scheme under the assumption that all system parameters were known, in order to illustrate the fundamental control structure. Subsequently, we present an adaptive version of the nominal control structure to address parameter uncertainties within the agents. Finally, we devote to design a modified MRAC law to handle the complete uncertainty associated with the control gain parameter k_{p_i} in the follower agent's model (2.2). Stability analysis shows the stability of closed-loop system and the desirable output tracking performance with the developed distributed adaptive control schemes.

3.1. Nominal control design: Known parameter case. First, we design a nominal distributed MRC scheme for the MAS consisting of (2.1) and (2.2) with all system parameters being known.

Nominal controller structure. Suppose all system parameters $A_i, b_i, c_i,\ i = 0, 1, ..., N$, are prior known, a unified nominal distributed control law for each agent $i,\ i = 1, ..., N$, is designed as

$$u_i(k) = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} \left(\theta_{i1}^{*T} x_i(k) + \theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^{*} u_j(k) \right), \tag{3.1}$$

where $\theta_{i1}^* \in \mathbb{R}^{n_i}, \theta_{ij1}^* \in \mathbb{R}^{n_j}, \theta_{ij2}^* \in \mathbb{R}$ are some constant parameters to be determined by utilizing agents' system parameters, $x_i(k)$ is the measurable state variable of agent i, and $x_j(k)$ and $u_j(k)$ are the state and input of neighboring agent $j \in \mathcal{N}_i$. Moreover, d_i is the indegree of the agent i, i.e., the total agent number in the neighbor set $\mathcal{N}_i$.

Now, we give the details about calculating the controller parameters $\theta_{i1}^*, \theta_{ij1}^*, \theta_{ij2}^*$ in (3.1) for $i = 1, ..., N$. First, for $i = 1, ..., N$, there exist some constant parameters θ_{i1}^* and θ_{i2}^* such that

$$\det(zI - A_i - b_i \theta_{i1}^{*T}) = z^{n^*} Z_i(z) \theta_{i2}^*,\ \theta_{i2}^* = \frac{1}{k_{p_i}}. \tag{3.2}$$

This existence of θ_{i1}^* and θ_{i2}^* is ensured by Assumptions (A1) and (A2). Readers may refer to the monograph [27] for details on single-agent systems. Thus, the controller parameters $\theta_{i1}^*,\ i = 1, ..., N$, can be calculated from the equation (3.2).

Since each follower agent and the leader agent have the relative degree n^*, this implies that $c_i A_i^j b_i = 0$ for $j = 1, ..., n^* - 2,\ i = 0, 1, ..., N$, and $c_i A_i^{n^*-1} b_i \neq 0$ for $i = 0, 1, ..., N$. Thus, motivated by the idea in [41, 42], using the definition of system relative degree, for $i = 0, 1, ..., N$, we obtain

$$y_i(k+j) = \begin{cases} c_i A_i^j x_i(k), & j = 0, ..., n^* - 1, \\ c_i A_i^j x_i(k) + c_i A_i^{j-1} b_i u_i(k), & j = n^*. \end{cases}$$

Therefore, the system future time output signal $y_i(k + n^*)$ can be expressed by $x_i(k), u_i(k)$ as

$$y_i(k + n^*) = \alpha_{i1}^{*T} x_i(k) + \alpha_{i2}^* u_i(k),\ i = 0, 1, ..., N, \tag{3.3}$$

where $\alpha_{i1}^* \in \mathbb{R}^{n_i}$ and $\alpha_{i2}^* \in \mathbb{R}$ are constant parameters related to the system parameters (A_i, b_i, c_i). With $\alpha_{i1}^*, \alpha_{i2}^*$ in (3.3) and θ_{i2}^* in (3.2), the controller parameters θ_{ij1}^* and θ_{ij2}^* are determined by

$$\theta_{ij1}^* = \theta_{i2}^* \alpha_{j1}^*,\ \theta_{ij2}^* = \theta_{i2}^* \alpha_{j2}^*,\ i = 1, ..., N,\ j \in \mathcal{N}_i. \tag{3.4}$$

So far, all nominal controller parameters in (3.1) are determined based on the equations (3.2) and (3.3) assuming that all system parameters are known.

Remark 3.1. From (3.1), one can see that the computation of each control input $u_i(k)$ depends on its neighboring control input $u_j(k)$. This information exchange enables the construction of a virtual reference signal for the distributed consensus tracking control design when the leader is non-autonomous. Based on Assumption (A4), all agents in the MAS are ordered such that A is upper triangular. Consequently, the control input $u_1(k)$ can be determined first using the reference input $u_0(k)$, enabling the sequential computation of all subsequent control inputs from agent 1 to N. In other words, all control inputs are well-defined under the graph structure, which is adopted in some prior literature [30, 33, 35, 31]. Such a sequential dependency is reasonable in certain MAS scenarios, where agents make their control actions based on information exchanged with their neighbors. Since each agent exchanges its state vector and a scalar control input with its neighbors, this design increases the communication burden. How to reduce the communication burden within the proposed framework is an important issue and will be considered in our future research

Remark 3.2. Based on (3.3), the signal $y_i(k + n^*)$ can be expressed in terms of its own parameter information, state vector, and control input. By exchanging state vectors, control inputs, and parameter information with neighbors, the controller (3.1) can be implemented without requiring knowledge of the leader's information. Therefore, the nominal controller (3.1) is distributed since it relies only on local neighboring information. For systems with unknown parameters, the expression (3.3) serves as a parameterized form and plays a critical role in the adaptive parameter estimation, as will be demonstrated in the following sections.

Remark 3.3. In this work, we regard the interaction graph $\mathcal{G}$ of the MAS (2.1) and (2.4) is unweighted. That is, the link weight a_{ij} is set to 1 if agent i could receive agent j's information. Therefore, the weight a_{ij} could be omitted in the nominal controller (3.1). However, to depict the varying influences among individuals, the link weight a_{ij} can be assigned distinct positive values. In this weighted graph case, the information from neighbors needs to be multiplied by the communication link weight to design the controller for each follower agent $i,\ i = 1, ..., N$. A feasible control law in this situation takes the following form:

$$u_i(k) = \frac{1}{\sum_{j\in\mathcal{N}_i} a_{ij}} \sum_{j\in\mathcal{N}_i} a_{ij} \left(\theta_{i1}^{*T} x_i(k) + \theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^* u_j(k) \right).$$

The adjustment for the weighted graph case follows an idea similar to the continuous-time distributed MRAC law studied in [32, 33]. The control strategy and design methodology proposed in this study may be generalized to address the weighted graph case by incorporating appropriate weighted parameters a_{ij}. Such an extension would entail a significant increase in both computational effort and system complexity, which needs further investigation. In this paper, we assume unit weights $(a_{ij} = 1)$ in the formulation of the distributed MRAC framework.

Stability analysis. Next, we proceed to analyze the system performance of the MAS under the nominal distributed control law (3.1). To this end, we formally define the local output tracking error and the leader-following tracking error as follows

$$e_i(k) = y_i(k) - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} y_j(k),\ i = 1, ..., N, \tag{3.5}$$

$$e_{i0}(k) = y_i(k) - y_0(k),\ i = 1, ..., N, \tag{3.6}$$

respectively. The local output tracking error $e_i(k)$ quantifies the deviation between follower agent i's output and its neighbors' average output. The leader-following tracking error $e_{i0}(k)$ characterizes the discrepancy of system output between the follower agent and the leader. In the following lemma, a crucial property is shown to clarify the relationship between $e_i(k)$ and $e_{i0}(k)$.

LEMMA 3.4. *For the considered MAS comprising of N follower agents and one leader agent under the communication graph $\mathcal{G}$, if the local tracking error $e_i(k)$ defined in (3.5) satisfies $\lim_{k\to\infty} e_i(k) = 0$ (or $e_i(k) = 0$), then the leader-following tracking error would satisfy $\lim_{k\to\infty} e_{i0}(k) = 0$ (or $e_{i0}(k) = 0$) for $i = 1, ..., N$.*

Proof. The proof of Lemma 3.4 closely follows those of Lemma 4.1 in [30] and Lemma 3.6 in [34]. Therefore, it is omitted here for brevity. □

From this lemma, we see that the leader-following consensus tracking objective is achieved if all local output tracking errors satisfy $\lim_{k\to\infty} e_i(k) = 0$. Conversely, $\lim_{k\to\infty} e_i(k) = 0$ would be obvious if $\lim_{k\to\infty} e_{i0}(k) = 0$. Thus, Lemma 3.4 shows $\lim_{k\to\infty} e_i(k) = 0$ is necessary and sufficient for $\lim_{k\to\infty} e_{i0}(k) = 0$. Based on this observation, the following theorem establishes the closed-loop signal boundedness and characterizes the leader-following tracking performance achieved by the nominal distributed controller (3.1).

THEOREM 3.5. *Under Assumptions (A1)-(A4), the nominal distributed controller (3.1) ensures that all closed-loop signals are bounded and the output of each follower agent $y_i(k)$ precisely tracks the output of leader agent $y_0(k)$ after n^* sampling periods, that is, $e_{i0}(k + n^*) = 0,\ i = 1, ..., N$.*

Proof. For each agent $i,\ i = 1, ..., N$, substituting the controller (3.1) into the state equation of the system model (2.1), we obtain

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + \frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{i1}^{*T} x_i(k) + \theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^{*} u_j(k)\right) \\ &= (A_i + b_i\theta_{i1}^{*T}) x_i(k) + \frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^{*} u_j(k)\right). \end{aligned} \tag{3.7}$$

From the matching equation (3.2), it yields

$$c_i^T(zI - A_i - b_i\theta_{i1}^{*T})^{-1} b_i\theta_{i2}^{*} = \frac{Z_i(z)\theta_{i2}^{*}}{\det(zI - A_i - b_i\theta_{i1}^{*T})} = \frac{1}{z^{n^*}},\ i = 1, ..., N. \tag{3.8}$$

From (3.3) and (3.4), we note that $\theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^{*} u_j(k) = \theta_{i2}^{*} y_j(k + n^*)$ for $i = 1, ..., N,\ j \in \mathcal{N}_i$. Thus, it follows from (3.7), (3.8) and the system model (2.1) that

$$\begin{aligned} y_i(k+n^*) =& z^{n^*} c_i^T(zI - A_i - b_i\theta_{i1}^{*T})^{-1}\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{ij1}^{*T} x_j(k) + \theta_{ij2}^{*} u_j(k)\right) \\ =& \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} y_j(k+n^*),\ i = 1, ..., N, \end{aligned}$$

which implies the local output tracking error is always zero after n^* sampling periods, that is, $e_i(k+n^*)=0,\ i=1,...,N$. Then, it follows from the conclusion of Lemma 3.4 that the leader-following tracking error satisfies $e_{i0}(k+n^*)=0,\ i=1,...,N$.

Since the leader agent model (2.4) is stable and driven by a bounded external reference input signal $u_0(k)$, its state variable $x_0(k)$ and system output signal $y_0(k)$ are all bounded. This also demonstrates that the follower agent's output signal $y_i(k),\ i=1,...,N$, is bounded given that $e_{i0}(k+n^*)=0$. Using the system model (2.2), we have $k_{p_i}z^{n^*}Z_i(z)[u_i](k)=P_i(z)[y_i](k+n^*),\ i=1,...,N$, which demonstrates that $u_i(k)\in L^\infty$ with the boundedness of $y_i(k)$ since $z^{n^*}Z_i(z)$ is stable and of degree $n^*+m_i=n_i$. Furthermore, considering the system model (2.1) is detectable under Assumption (A1), the unobservable state variable of follower agent i is stable and the observable part of $x_i(k)$ could be constructed by its input signal $u_i(k)$ and output signal $y_i(k)$. Together with the bounedness of $u_i(k)$ and $y_i(k)$, we obtain $x_i(k)\in L^\infty$. Therefore, all signals in the closed-loop MAS are bounded. This completes the proof.□

Now, a nominal MRC scheme has been established, incorporating a unified distributed control law (3.1) for each follower agent i. Given all system parameters were known, this control law could achieve precise leader-following output tracking by using only the neighboring signals $x_j(k)$ and $u_j(k),\ j\in\mathcal{N}_i$.

3.2. Distributed MRAC design: Unknown parameter case. This subsection addresses the main focus of the paper, in which all system parameters, including those of both leader and follower agents, are unknown. In this situation, the nominal distributed control law (3.1) is no longer applicable due to the presence of system uncertainties. Now, we consider to develop an adaptive version of the nominal control scheme for achieving the control objective of the paper.

Distributed MRAC law. Based on the nominal distributed control law (3.1), for each follower agent $i,\ i=1,...,N$, a distributed MRAC law is designed as follows

$$u_i(k)=\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{i1}^T(k)x_i(k)+\theta_{ij1}^T(k)x_j(k)+\theta_{ij2}(k)u_j(k)\right), \tag{3.9}$$

where $\theta_{i1}(k),\theta_{ij1}(k),\theta_{ij2}(k)$ are estimates of $\theta_{i1}^*,\theta_{i21}^*,\theta_{i22}^*$, respectively. Similarly, the involved signals for the adaptive control law (3.9) of agent i includes its state vector $x_i(k)$, and its neighboring agents' state vector $x_j(k)$ and control input signal $u_j(k)$, which could be obtained by local communication in the graph $\mathcal{G}$.

Tracking error equation. To adaptively update the controller parameter estimates $\theta_{i1}(k),\theta_{ij1}(k),\theta_{ij2}(k)$ for each agent i, we first construct a tracking error equation for the local output tracking error $e_i(k),\ i=1,...,N$. Combing the adaptive control law (3.9) and the follower agent dynamic model (2.1) for each agent $i,\ i=1,...,N$, we have

$$\begin{aligned}
x_i(k+1) &= A_ix_i(k)+\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{i1}^T(k)x_i(k)+\theta_{ij1}^T(k)x_j(k)+\theta_{ij2}(k)u_j(k)\right)\\
&= (A_i+b_i\theta_{i1}^{*T})x_i(k)+\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{ij1}^{*T}x_j(k)+\theta_{ij2}^*u_j(k)\right)\\
&\quad +b_i\left(\theta_{i1}^T(k)-\theta_{i1}^{*T}\right)x_i(k)+\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{ij1}^T(k)-\theta_{ij1}^{*T}\right)x_j(k)\\
&\quad +\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left(\theta_{ij2}(k)-\theta_{ij2}^*\right)u_j(k).
\end{aligned}$$

Then, it follows from (2.1) and (3.8) that

$$\begin{aligned} y_i(k) \,=\,& c_i^T(zI-A_i-b_i\theta_{i1}^{*T})^{-1}\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left[\theta_{ij1}^{*T}x_j+\theta_{ij2}^{*}u_j\right](k) \\ &+c_i^T(zI-A_i-b_i\theta_{i1}^{*T})^{-1}b_i\left[(\theta_{i1}^{T}-\theta_{i1}^{*T})x_i\right](k) \\ &+c_i^T(zI-A_i-b_i\theta_{i1}^{*T})^{-1}\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left[(\theta_{ij1}^{T}-\theta_{ij1}^{*T})x_j\right](k) \\ &+c_i^T(zI-A_i-b_i\theta_{i1}^{*T})^{-1}\frac{b_i}{d_i}\sum_{j\in\mathcal{N}_i}\left[(\theta_{ij2}-\theta_{ij2}^{*})u_j\right](k) \\ =\,& \frac{1}{d_i}\sum_{j\in\mathcal{N}_j}y_j(k)+\frac{\rho_i^*}{z^{n^*}}\left[(\theta_{i1}^{T}-\theta_{i1}^{*T})x_i\right](k) \\ &+\frac{\rho_i^*}{z^{n^*}}\left[\sum_{j\in\mathcal{N}_i}(\theta_{ij1}^{T}-\theta_{ij1}^{*T})\frac{x_j}{d_i}\right](k)+\frac{\rho_i^*}{z^{n^*}}\left[\sum_{j\in\mathcal{N}_i}(\theta_{ij2}-\theta_{ij2}^{*})\frac{u_j}{d_i}\right](k), \end{aligned}$$

where $\rho_i^*\triangleq k_{p_i},\ i=1,...,N$. Thus, we obtain the tracking error equation for $e_i(k)$ defined in (3.5) as

$$\begin{aligned} e_i(k) \,=\,& \frac{\rho_i^*}{z^{n^*}}\left[(\theta_{i1}^{T}-\theta_{i1}^{*T})x_i\right](k)+\frac{\rho_i^*}{z^{n^*}}\left[\sum_{j\in\mathcal{N}_i}(\theta_{ij1}^{T}-\theta_{ij1}^{*T})\frac{x_j}{d_i}\right](k) \\ &+\frac{\rho_i^*}{z^{n^*}}\left[\sum_{j\in\mathcal{N}_i}(\theta_{ij2}-\theta_{ij2}^{*})\frac{u_j}{d_i}\right](k),\ i=1,...,N. \end{aligned} \tag{3.10}$$

This is the desired tracking error equation for the local output tracking error $e_i(k)$.

Estimation error equation. Next, we introduce an estimation error to measure the difference between nominal values of controller parameters and their estimates. For $i=1,...,N$, we define

$$\begin{aligned} \theta_i(k) \,\triangleq\,& \left[\theta_{i1}^{T}(k),\theta_{ij1}^{T}(k)|_{j\in\mathcal{N}_i},\theta_{ij2}(k)|_{j\in\mathcal{N}_i}\right]^T, \\ \theta_i^* \,\triangleq\,& \left[\theta_{i1}^{*T},\theta_{ij1}^{*T}|_{j\in\mathcal{N}_i},\theta_{ij2}^{*}|_{j\in\mathcal{N}_i}\right]^T,\ \tilde{\theta}_i(k)\triangleq\theta_i(k)-\theta_i^*, \end{aligned} \tag{3.11}$$

and introduce the following auxiliary signals

$$\begin{aligned} \omega_i(k) \,\triangleq\,& \left[x_i^T(k),\frac{x_j^T(k)}{d_i}|_{j\in\mathcal{N}_i},\frac{u_j(k)}{d_i}|_{j\in\mathcal{N}_i}\right]^T, \\ \bar{\omega}_i(k) \,\triangleq\,& \frac{1}{z^{n^*}}\left[\omega_i\right](k),\ \xi_i(k)\triangleq\theta_i(k)\bar{\omega}_i(k)-\frac{1}{z^{n^*}}\left[\theta_i\omega_i\right](k). \end{aligned} \tag{3.12}$$

where the notation $\theta_{ij1}^{*T}|_{j\in\mathcal{N}_i}$ represents the combination of parameter vector θ_{ij1}^{*T} in a specific order of j for $j\in\mathcal{N}_i$, so are the following similar notation. With definitions in (3.11) and (3.12), the tracking error equation (3.10) could be rewritten into the following compact form

$$e_i(k)=\frac{\rho_i^*}{z^{n^*}}\left[\tilde{\theta}_i^T\omega_i\right](k),\ i=1,...,N. \tag{3.13}$$

Then, an estimation error $\epsilon_i(k)$ is defined as

$$\epsilon_i(k) = e_i(k) + \rho_i(k)\xi_i(k),\ i = 1, ..., N. \tag{3.14}$$

where $\rho_i(k)$ is an estimate of ρ_i^*. Combining the tracking error equation (3.10) and the estimation error (3.14), we obtain

$$\epsilon_i(k) = \rho_i^* \tilde{\theta}_i^T(k)\bar{\omega}_i(k) + \tilde{\rho}_i(k)\xi_i(k),\ i = 1, ..., N, \tag{3.15}$$

where $\tilde{\rho}_i(k) \triangleq \rho_i(k) - \rho_i^*$. This is actually the estimation error equation, which measures the discrepancy between nominal controller parameters and the estimated ones.

Parameter update law. Now, we develop an adaptive parameter update law for the estimated controller parameters in (3.9) with estimation error cost criterion. Before this, we need to introduce an assumption to facilitate the following design.

(A5) For each agent $i,\ i = 1, ..., N$, the sign of k_{p_i}, i.e., sign$[k_{p_i}]$, is known, and an upper bound $k_{p_i}^0$ of $|k_{p_i}|$, i.e., $k_{p_i}^0 \geq |k_{p_i}|$, is also known.

Assumption (A5) shows certain prior knowledge of high-frequency gain k_{p_i} of agent i is required, which is necessary for the following parameter update law design under the distributed MRAC law (3.9).

Introduce a quadratic cost function $J_i = \epsilon_i^2/2m_i^2$ for each agent $i,\ i = 1, ..., N$, with the defined estimation error $\epsilon_i(k)$ in (3.14), where $m_i = m_i(k)$ is a normalization signal defined as

$$m_i(k) = \sqrt{1 + \bar{\omega}_i^T(k)\bar{\omega}_i(k) + \xi_i^2(k)}. \tag{3.16}$$

Then, we compute the gradients of J_i with respect to $\theta_i(k)$ and $\rho_i(k)$ as $\frac{\partial J_i}{\partial \theta_i} = \frac{\rho_i^* \epsilon_i(k)\bar{\omega}_i(k)}{m_i^2(k)}$, $\frac{\partial J_i}{\partial \rho_i} = \frac{\epsilon_i(k)\xi_i(k)}{m_i^2(k)}$. Updating parameter estimates $\theta_i(k)$ and $\rho_i(k)$ along the negative gradient direction of J_i, we develop a parameter update law as

$$\begin{aligned} \theta_i(k+1) &= \theta_i(k) - \frac{\text{sign}[k_{p_i}]\Gamma_i \epsilon_i(k)\bar{\omega}_i(k)}{m_i^2(k)},\ i = 1, ..., N, \\ \rho_i(k+1) &= \rho_i(k) - \frac{\gamma_i \epsilon_i(k)\xi_i(k)}{m_i^2(k)},\ i = 1, ..., N, \end{aligned} \tag{3.17}$$

where $0 < \Gamma_i = \Gamma_i^T < 2I/k_{p_i}^0$ and $0 < \gamma_i < 2$ are constant adaptive gains to be chosen by designers. Now, we have designed a gradient based parameter update law (3.17) for each follower agent $i,\ i = 1, ..., N$, to update parameter estimates $\theta_i(k)$ and $\rho_i(k)$ adaptively.

Remark 3.6. The parameter update law (3.17) requires the sign information and bound knowledge of k_{p_i}, as specified in Assumption (A5). To be specific, sign$[k_{p_i}]$ determines the update direction of the parameter estimate $\theta_i(k)$, while the bound $k_{p_i}^0$ sets the adaptive gain to ensure the stability of the parameter update law (3.17). This requirement originates from the fact that the estimation error equation (3.15) exhibits bilinear regression characteristics with respect to the parameter estimate $\theta_i(k)$ under the adaptive control law (3.9). Because of the explicit multiplication of ρ_i^* and $\tilde{\theta}_i(k)$ in (3.10), it is necessary to introduce sign information and bound knowledge of k_{p_i} for stable parameter update law derivation.

Remark 3.7. Compared with the indirect distributed MRAC method in [34], the proposed distributed MRAC law has the following advantages: (i) The computational complexity is lower since the controller parameters are directly updated by the adaptive law (3.17) without solving a Diophantine equation online; (ii) The design of the controller (3.9) does not rely on the parameter information of the leader system, and thus can handle parameter uncertainties in all agents; (iii) The controller structure (3.9) is unified for all follower agents, and thus there is no need to classify followers according to graph connectivity.

The adaptive parameter update law (3.17) has the following properties.

LEMMA 3.8. *For each agent i, $i = 1, ..., N$, the proposed adaptive parameter update law (3.17) ensures that $\theta_i(k) \in L^\infty$, $\rho_i(k) \in L^\infty$, $\frac{\epsilon_i(k)}{m_i(k)} \in L^2 \cap L^\infty$, $\theta_i(k+1) - \theta_i(k) \in L^2 \cap L^\infty$, and $\rho_i(k+1) - \rho_i(k) \in L^2 \cap L^\infty$.*

Proof. Choose positive definite function: $V_i(\tilde{\theta}_i, \tilde{\rho}_i) = |\rho_i^*|\tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i + \gamma_i^{-1} \tilde{\rho}_i{}^2$. Since $0 < \Gamma_i = \Gamma_i^T < 2I/k_{p_i}^0$, $k_{p_i}^0 \geq |k_{p_i}| = |\rho_i^*|$ and $0 < \gamma_i < 2$, the time increment of $V(\tilde{\theta}_i, \tilde{\rho}_i)$ along the trajectories of parameter update law (3.17) satisfies

$$
\begin{aligned}
& V_i(\tilde{\theta}_i(k+1), \tilde{\rho}_i(k+1)) - V_i(\tilde{\theta}_i(k), \tilde{\rho}_i(k)) \\
&= -\left(2 - \frac{|k_{p_i}|\bar{\omega}_i^T(k)\Gamma_i\bar{\omega}_i(k) + \gamma_i \xi_i^2(k)}{m_i^2(k)}\right)\frac{\epsilon_i^2(k)}{m_i^2(k)} \leq -\nu_i \frac{\epsilon_i^2(k)}{m_i^2(k)}
\end{aligned}
$$

for some constant $\nu_i > 0$. This implies that $\theta_i(k) \in L^\infty, \rho_i(k) \in L^\infty$, and $\frac{\epsilon_i(k)}{m_i(k)} \in L^2$. Then, from the equation (3.15) and the definition of $m_i(k)$ in (3.16), we have $\frac{\epsilon_i(k)}{m_i(k)} \in L^\infty$. Finally, with the adaptive parameter update law (3.17), we obtain $\frac{\epsilon_i(k)}{m_i(k)} \in L^2 \cap L^\infty$, $\theta_i(k+1) - \theta_i(k) \in L^2 \cap L^\infty$, and $\rho_i(k+1) - \rho_i(k) \in L^2 \cap L^\infty$. This completes the proof. □

Stability analysis. We now analyse the control performance achieved by the distributed MRAC law (3.9) and parameter update law (3.17), which is specified in the following theorem.

THEOREM 3.9. *For the MAS (2.1) and (2.4) with communication graph $\mathcal{G}$, under Assumptions (A1)-(A5), if the adaptive control law (3.9) along with the parameter update law (3.17) is applied to each follower agent i, then the closed-loop system is stable and the output $y_i(k)$ of each follower agent tracks the leader's output signal $y_0(k)$ asymptotically, i.e., $\lim_{k\to\infty} e_{i0}(k) = 0$, $i = 1, ..., N$.*

Proof. See Appendix. □

So far, we have developed a distributed MRAC scheme for the discrete-time uncertain MAS (2.1) and (2.4). The distributed MRAC law (3.9) with the gradient based parameter update law (3.17) achieves leader-following asymptotical tracking using only local neighboring information. Notably, stable implementation of parameter update law (3.17) requires prior knowledge of the sign and bound of control gain k_{p_i} as specified in Assumption (A5).

3.3. Modified distributed MRAC design: Without prior knowledge of control gain. The uncertainty associated with high-frequency gain parameter presents a fundamental challenge in conventional MRAC of single-agent systems. While assuming known gain sign is a common technique for stable adaptive laws, practical applications often cannot satisfy this condition [43]. Therefore, it is desirable to eliminate this constraint on prior information of high-frequency gain parameter. Similarly, for uncertain MASs, their adaptive control designs encounter the same limitation. As demonstrated earlier, Assumption (A5) remains necessary for the parameter update law (3.17) under the adaptive control law (3.9). In this subsection, we further focus on designing a modified distributed MRAC law without relying on the additional Assumption (A5), while still achieving the control objective.

A modified distributed MRAC law. With θ_i^* defined in (3.11) and $\rho_i^* \triangleq k_{p_i}$, we first define β_i^* as $\beta_i^* \triangleq \rho_i^* \theta_i^*$ for $i = 1, ..., N$. Then, for each agent i, a modified distributed MRAC law is designed as

$$u_i(k) = (1 + \kappa_i(k)\rho_i(k))^{-1} \left(\theta_i^T(k)\omega_i(k) + \kappa_i(k)\beta_i^T(k)\omega_i(k)\right), \tag{3.18}$$

where $\theta_i(k), \beta_i(k), \rho_i(k)$ are the estimates of $\theta_i^*, \beta_i^*, \rho_i^*$, respectively, and $\omega_i(k)$ is regressor vector defined in (3.12), and $\kappa_i(k)$ denotes a time-varying gain function and will be specifically designed for each agent i to ensure $(1 + \kappa_i(k)\rho_i(k))^{-1}$ remains non-zero in the control process.

Remark 3.10. Different from the adaptive control law (3.9), we design a modified distributed MRAC law (3.18) by introducing an extra parameter estimates $\beta_i(k)$, which implicitly estimates the control gain k_{p_i}. The purpose of this modification is to compensate for the uncertainty of the control gain through an extra adaptive parameter. As will be shown below, the resulting estimation error model admits a linear regression form, which eliminates the need for prior sign information of k_{p_i} in the parameter update law. Under the ideal case, i.e., $\theta_i(k) = \theta_i^*$, $\beta_i(k) = \beta_i^*$, and $\rho_i(k) = \rho_i^*$, the control law (3.18) reduces to the nominal controller (3.1). To ensure the validity of (3.18), a time-varying function $\kappa_i(k)$ is introduced. In the following, we show that with a proper design of $\kappa_i(k)$, (3.18) remains non-singular at all times.

Tracking error equation. Based on the adaptive control law (3.18), we now derive a new tracking error equation for the local output tracking error $e_i(k)$, $i = 1, ..., N$. First, for $i = 1, ..., N$, it follows from the dynamic model (2.1) that

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + b_i \left(\theta_{i1}^{*T} x_i(k) + \theta_{i2}^* \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} z^{n^*} [y_j](k)\right) + b_i \left(u_i(k) - \theta_i^{*T} \omega_i(k)\right) \\ &= \left(A_i + b_i \theta_{i1}^{*T}\right) x_i(k) + b_i^* \theta_{i2}^* \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} z^{n^*} [y_j](k) + b_i \left(u_i(k) - \theta_i^{*T} \omega_i(k)\right). \end{aligned}$$

Combining the matching equation (3.8), we obtain the output signal $y_i(k)$ satisfies

$$\begin{aligned} y_i(k) &= c_i^T \left(zI - A_i - b_i \theta_{i1}^{*T}\right)^{-1} b_i \theta_{i2}^* \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} z^{n^*} [y_j](k) \\ &\quad + c_i^T \left(zI - A_i - b_i \theta_{i1}^{*T}\right)^{-1} b_i \left[u_i - \theta_i^{*T} \omega_i\right](k) \\ &= \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} y_j(k) + \frac{\rho_i^*}{z^{n^*}} \left[u_i - \theta_i^{*T} \omega_i\right](k), \end{aligned}$$

which indicates that the local output tracking error $e_i(k)$ satisfies

$$e_i(k) = \frac{\rho_i^*}{z^{n^*}} \left[u_i - \theta_i^{*T} \omega_i\right](k), \ i = 1, ..., N. \tag{3.19}$$

This is a signal identity for any control input $u_i(k)$. Define $\mu_i^* \triangleq \frac{1}{\rho_i^*}$ and denote $\mu_i(k)$ being the estimate of μ_i^* for $i = 1, ..., N$. Then, it follows from (3.19) that

$$\mu_i^* e_i(k) = \frac{1}{z^{n^*}} \left[u_i - \theta_i^{*T} \omega_i\right](k), \ i = 1, ..., N. \tag{3.20}$$

From the adaptive control law (3.9) and the plant signal identity (3.19), we have

$$\begin{aligned}
u_i(k) =\ & \theta_i^T(k)\omega_i(k) + \kappa_i(k)\beta_i^T(k)\omega_i(k) - \alpha_i(k)\rho_i(k)u_i(k) \\
& +\alpha_i(k)z^{n^*}[e_i](k) - \kappa_i(k)z^{n^*}[e_i](k) \\
=\ & \theta_i^T(k)\omega_i(k) + \kappa_i(k)(\beta_i^T(k)\omega_i(k) - \rho_i(k)u_i(k)) \\
& +\kappa_i(k)(\rho_i^* u_i(k) - \beta_i^{*T}\omega_i(k)) - \kappa_i(k)z^{n^*}[e_i](k) \\
=\ & \theta_i^T(k)\omega_i(k) + \kappa_i(k)\tilde{\beta}_i^T(k)\omega_i(k) - \kappa_i(k)\tilde{\rho}_i(k)u_i(k) \\
& -\kappa_i(k)z^{n^*}[e_i](k),\ i = 1, ..., N,
\end{aligned} \tag{3.21}$$

where $\tilde{\beta}_i(k) \triangleq \beta_i(k) - \beta_i^*$ and $\tilde{\rho}_i(k) \triangleq \rho_i(k) - \rho_i^*$. Combining (3.20) and (3.21), we derive

$$(\mu_i^* + \kappa_i(k))\, z^{n^*}[e_i](k) = \tilde{\theta}_i^T(k)\omega_i(k) + \kappa_i(k)\tilde{\beta}_i^T(k)\omega_i(k) - \kappa_i(k)\tilde{\rho}_i(k)u_i(k),$$

where $\tilde{\theta}_i(k) \triangleq \theta_i(k) - \theta_i^*$. Denote $\mu_i(k)$ as the estimate of μ_i^* and define $\tilde{\mu}_i(k) \triangleq \mu_i(k) - \mu_i^*$. Then, for $i = 1, ..., N$, we obtain

$$\begin{aligned}
(\mu_i(k) + \kappa_i(k))\, z^{n^*}[e_i](k) =\ & \tilde{\rho}_i(k)z^{n^*}[e_i](k) + \tilde{\theta}_i^T(k)\omega_i(k) + \kappa_i(k)\tilde{\beta}_i^T(k)\omega_i(k) \\
& -\kappa_i(k)\tilde{\rho}_i(k)u_i(k).
\end{aligned}$$

Under the condition that $\mu_i(k) + \alpha_i(k) \neq 0$, for $i = 1, ..., N$, it yields

$$\begin{aligned}
z^{n^*}[e_i](k) =\ & \tilde{\mu}_i(k)\frac{z^{n^*}[e_i](k)}{\mu_i(k) + \kappa_i(k)} + \tilde{\theta}_i^T(k)\frac{\omega_i(k)}{\mu_i(k) + \kappa_i(k)} \\
& +\tilde{\beta}_i^T(k)\frac{\kappa_i(k)\omega_i(k)}{\mu_i(k) + \kappa_i(k)} - \tilde{\rho}_i(k)\frac{\kappa_i(k)u_i(k)}{\mu_i(k) + \kappa_i(k)}.
\end{aligned} \tag{3.22}$$

For the sake of convenience, we introduce the following parameter vectors

$$\begin{aligned}
\lambda_i(k) \triangleq\ & [\mu_i(k), \theta_i^T(k), \beta_i^T(k), \rho_i(k)]^T, \\
\lambda_i^* \triangleq\ & [\mu_i^*, \theta_i^{*T}, \beta_i^{*T}, \rho_i^*]^T,\ \tilde{\lambda}_i(k) \triangleq \lambda_i(k) - \lambda_i^*,
\end{aligned} \tag{3.23}$$

and the regressor signal

$$\psi_i(k) \triangleq \left[\frac{z^{n^*}[e_i](k)}{\mu_i(k) + \kappa_i(k)}, \frac{\omega_i^T(k)}{\mu_i(k) + \kappa_i(k)}, \frac{\kappa_i(k)\omega_i^T(k)}{\mu_i(k) + \kappa_i(k)}, -\frac{\kappa_i(k)u_i(k)}{\mu_i(k) + \kappa_i(k)}\right]^T. \tag{3.25}$$

Finally, with definitions in (3.23) and (3.25), the equation (3.22) could be transformed as

$$z^{n^*}[e_i](k) = \tilde{\lambda}_i^T(k)\psi_i(k),\ i = 1, ..., N. \tag{3.26}$$

Now we get a closed-loop tracking error equation (3.26) for the local output tracking error $e_i(k)$ under the proposed distributed adaptive control law (3.18).

Remark 3.11. Motivated by the signal identity (3.19), the nominal control law $u_i(k) = \theta_i^T\omega_i(k)$ ensures zero local tracking error $e_i(k)$ after n^* sampling periods when θ_i^* is known. Replacing θ_i^* with its estimate $\theta_i(k)$ yields the certainty equivalence adaptive control law $u_i(k) = \theta_i^T(k)\omega_i(k)$, i.e., the distributed MRAC law (3.9). Its validity, however, requires the strict prior knowledge of the control gain k_{p_i} as stated in Assumption (A5).

Parameter update law and singularity-free design. We now derive a gradient based parameter update law for controller parameters in (3.18) by using an estimation error cost criterion. For each agent $i,\ i=1,...,N$, define an estimation error $\epsilon_i(k)$ as

$$\epsilon_i(k) = e_i(k) + \lambda_i^T(k)\frac{1}{z^{n^*}}[\psi_i](k) - \frac{1}{z^{n^*}}[\lambda_i^T\psi_i](k). \tag{3.27}$$

Then, it can be clearly observed that the signal $\epsilon_i(k)$ is available at the current time instant for each agent $i,\ i=1,...,N$. Combining the tracking error equation (3.26) and the estimation error (3.27), we derive that

$$\epsilon_i(k) = \tilde{\lambda}_i^T(k)\bar{\psi}_i(k),\ i=1,...,N, \tag{3.28}$$

where $\bar{\psi}_i(k) \triangleq \frac{1}{z^{n^*}}[\psi_i](k)$. This is the estimation error equation desired for the design of parameter adaptation.

Based on the defined estimation error (3.27), for each agent i, we introduce a quadratic cost function as $J_i = \epsilon_i^2/2\bar{m}_i^2$, where $\bar{m}_i = \bar{m}_i(k)$ is a normalization signal defined as

$$\bar{m}_i(k) = \sqrt{1 + \bar{\psi}_i^T(k)\bar{\psi}_i(k)}. \tag{3.29}$$

Then, we derive the gradient of J_i with respect to $\lambda_i(k)$ as $\frac{\partial J_i}{\partial \lambda_i} = \frac{\epsilon_i(k)\bar{\psi}_i(k)}{\bar{m}_i^2(k)}$, which motivates a gradient based parameter update law for $\lambda_i(k)$ as

$$\lambda_i(k+1) = \lambda_i(k) - \frac{\Upsilon_i\epsilon_i(k)\bar{\psi}_i(k)}{\bar{m}_i^2(k)},\ i=1,...,N, \tag{3.29}$$

with $0 < \Upsilon_i = \Upsilon_i^T < 2I$ being the constant adaptive gain to be designed.

Remark 3.12. To prevent blowup of the regressor $\psi_i(k)$, $\mu_i(k)+\kappa_i(k)$ must remain nonzero during parameter adaptation, with $\mu_i(k)$ updated by (3.29) and $\kappa_i(k)$ being a designed gain function. Similarly, in the distributed adaptive law (3.18), $1+\kappa_i(k)\rho_i(k)$ must stay nonzero with $\rho_i(k)$ updated by (3.29). In the following, we design $\kappa_i(k)$ to guarantee both conditions for all agents $i=1,...,N$.

Remark 3.13. The modified parameter update law (3.29) differs from (3.17) in two key aspects: (i) it does not require prior knowledge of the sign of each agent's uncertain high-frequency gain k_{p_i}, and (ii) the adaptive gain Υ_i can be chosen without knowledge of the gain bounds. Consequently, the modified MRAC law (3.18) no longer relies on Assumption (A5), relaxing the design requirements. The controller introduces an adaptive parameter β_i^* that implicitly embeds k_{p_i}, yielding a linear regressive estimation error (3.28) rather than the bilinear form (3.15) where ρ_i^* multiplies $\tilde{\theta}_i(k)$ explicitly. This allows the parameter update to follow the negative gradient of J_i without sign information of k_{p_i}. Stability is ensured for Υ_i satisfying $0 < \Upsilon_i = \Upsilon_i^T < 2I$ without requiring gain bounds, as detailed in the subsequent analysis of (3.29). However, the cost of the modified distributed adaptive control law (3.18) is that the dimension of the estimated parameter vector $\lambda_i(k)$ becomes larger compared with the distributed MRAC law (3.17). Despite this, the proposed algorithm remains simple to implement and tune since the parameter update law (3.29) employs a common adaptation gain Υ_i.

Remark 3.14. Notably, the proposed adaptive control law (3.18) avoids using the Nussbaum gain technique [3, 19, 20, 21, 22, 23, 24], thereby mitigating potential performance oscillations commonly associated with such technique. In the discrete-time setting, a Nussbaum function $N(\cdot)$ satisfies

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} N(\theta(j)) = +\infty,\ \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} N(\theta(j) = -\infty,$$

which implies that the function value $N(\cdot)$ changes sign infinitely often. As a result, when the control input is designed in the form $u_i(k) = N_i(\theta_i(k))v_i(k)$ with $v_i(k)$ being the distributed MRAC law (3.9), the oscillatory nature of the Nussbaum function may lead to repeated sign changes and large variations in the control input during the parameter adaptation process, which can cause undesirable transient oscillations in the system response.

To guarantee singularity-free of the distributed adaptive control law (3.18) and the parameter update law (3.29), the following conditions need to be guaranteed

$$1 + \kappa_i(k)\rho_i(k) \neq 0,\ \mu_i(k) + \kappa_i(k) \neq 0,\ i = 1, ..., N. \tag{3.29}$$

For this purpose, we give the following lemma, which specifies the design of time-varying gain function $\kappa_i(k)$.

LEMMA 3.15. *For each agent i, $i = 1, ..., N$, if the time-varying gain function $\kappa_i(k)$ is designed as*

$$\kappa_i(k) = \begin{cases} -\left(|\mu_i(k)| + \underline{\kappa}_i\right), & \rho_i(k) < 0, \\ |\mu_i(k)| + \underline{\kappa}_i, & \rho_i(k) \geq 0, \end{cases}$$

with arbitrary chosen constant $\underline{\kappa}_i > 0$, then condition (3.29) always hold in the adaptive process for $\rho_i(k) \in \mathbb{R}$ and $\mu_i(k) \in \mathbb{R}$ with any possible values.

The proof of this lemma is established in the literature [39, 40]. This lemma provides a design mechanism for the time-varying gain function $\kappa_i(k)$ that guarantees the singularity-free operation throughout the adaptive control process.

Remark 3.16. Actually, the time-varying gain function $\kappa_i(k)$ designed in Lemma 3.13 ensures that the terms $\mu_i(k) + \kappa_i(k)$ and $1 + \kappa_i(k)\rho_i(k)$ remain away from zero. Specifically, with the definition of $\kappa_i(k)$, we have

$$\mu_i(k) + \kappa_i(k) = \begin{cases} -\underline{\kappa}_i, & \rho_i(k) < 0,\ \mu_i(k) \geq 0, \\ 2\mu_i(k) - \underline{\kappa}_i, & \rho_i(k) < 0,\ \mu_i(k) < 0, \\ 2\mu_i(k) + \underline{\kappa}_i, & \rho_i(k) \geq 0,\ \mu_i(k) \geq 0, \\ -\underline{\kappa}_i, & \rho_i(k) \geq 0,\ \mu_i(k) < 0, \end{cases}$$

and

$$1 + \kappa_i(k)\rho_i(k) = \begin{cases} 1 - |\mu_i(k)|\rho_i(k) - \underline{\kappa}_i\rho_i(k), & \rho_i(k) < 0, \\ 1 + |\mu_i(k)|\rho_i(k) + \underline{\kappa}_i\rho_i(k), & \rho_i(k) \geq 0. \end{cases}$$

Since $\underline{\kappa}_i$ is a positive constant, we get $|\mu_i(k) + \kappa_i(k)| \geq \underline{\kappa}_i$ and $1 + \kappa_i(k)\rho_i(k) \geq 1$ always hold throughout the adaptive process. Therefore, Lemma 3.13 guarantees that no high-gain issues arise in the distributed adaptive control law.

To facilitate the final stability result, we first establish the following lemma, which specifies some beneficial properties about the given parameter update law (3.29).

LEMMA 3.17. *For each agent i, $i = 1, ...N$, the developed parameter update law (3.29) ensures that* $\lambda_i(k) \in L^\infty$, $\frac{\epsilon_i(k)}{\bar{m}_i(k)} \in L^2 \cap L^\infty$, *and* $\lambda_i(k+1) - \lambda_i(k) \in L^2 \cap L^\infty$.

Proof. For each agent i, choose a positive definite function as $V_i(\tilde{\lambda}_i) = \tilde{\lambda}_i^T \Upsilon_i^{-1} \tilde{\lambda}_i$. Because $0 < \Upsilon_i = \Upsilon_i^T < 2I$, combing the definition of $\bar{m}_i(k)$ in (3.29) gives

$$V_i(\tilde{\lambda}_i(k+1)) - V(\tilde{\lambda}_i(k)) = -\left(2 - \frac{\bar{\psi}_i^T(k)\Upsilon_i\bar{\psi}_i(k)}{\bar{m}_i^2(k)}\right)\frac{\epsilon_i^2(k)}{\bar{m}_i^2(k)} \le -s_i\frac{\epsilon_i^2(k)}{\bar{m}_i^2(k)}$$

for some constant $s_i > 0,\ i = 1, ..., N$. This implies that $\lambda_i(k) \in L^\infty$ and $\frac{\epsilon_i(k)}{\bar{m}_i(k)} \in L^2$. Then, we get $\frac{\epsilon_i(k)}{\bar{m}_i(k)} \in L^\infty$ from the estimation error equation (3.28). Further, from the parameter update law (3.29), we have $\lambda_i(k+1) - \lambda_i(k) \in L^2 \cap L^\infty$. This completes the proof. ☐

Stability analysis. Finally, the following theorem characterizes the system performance under the proposed modified distributed MRAC law (3.18).

THEOREM 3.18. *For the MAS (2.1) and (2.4) with communication graph $\mathcal{G}$, under Assumptions (A1)-(A4), if the adaptive control law (3.18) along with the parameter update law (3.29) is applied to each follower agent i, then the closed-loop system is stable and the output of each follower agent tracks the leader's output signal asymptotically, i.e.,* $\lim_{k\to\infty} e_{i0}(k) = 0,\ i = 1, ..., N$.

Proof. See Appendix. ☐

Remark 3.19. While recent literature has made significant strides in handling unknown high-frequency gains [39, 40] and reference uncertainties [41, 42] for single-agent systems, this paper tackles the consensus tracking control problem within an MASs framework. This extension is non-trivial due to some distributed control challenges. First, unlike single-agent systems with global reference access, the leader's information here is only available to a subset of agents. We address this by constructing a virtual reference signal based on neighbors' parameterized estimates of future outputs. Second, stability analysis is more complex as tracking and estimation errors are coupled across the network. By defining a local neighborhood tracking error and exploiting its topological relationship with the global leader-following tracking error in Lemma 3.4, we establish the stability of the entire MASs. In this framework, Assumption (A4) is key to ensure the well-posedness of the sequential adaptive design and managing the error propagation across the communication network.

So far, a new distributed MRAC scheme has been developed to achieve the leader-following tracking objective under Assumptions (A1)–(A4). The proposed control law implicitly estimates the control gain parameter k_{p_i}, leading to a linear regressive estimation error as described in (3.28). Based on this structure, a gradient-based parameter update law (3.29) is derived, enabling adaptive tuning without requiring prior knowledge of uncertain system parameters. This scheme is fundamentally different from the update law in (3.17).

In summary, a distributed MRAC framework has been developed for discrete-time linear time-invariant uncertain MASs, which comprises a nominal distributed MRC scheme formulated in Subsection 3.1, a fundamental distributed MRAC scheme formulated in Subsection 3.2, and a modified distributed MRAC scheme formulated in Subsection 3.3.

4. Simulation study. This section demonstrates the validity of the developed distributed MRAC schemes through a simulation case.

4.1. Simulation for distributed MRAC. This subsection first evaluates the control performance under the distributed MRAC law (3.9) and the parameter update law (3.17).

Simulation model. Consider an MAS consisting of one leader agent with the system model (2.4) and four follower agents with the system model (2.1). Specifically, the four follower agents' system parameter matrices are as follows

$$
\begin{aligned}
A_1 &= [0,1,0;0,0,1;1,2.5,0.5]\,, b_1 = [0,0,1]^T\,, c_1 = [0,0.5,1]^T\,;\\
A_2 &= [0,1;1,0]\,, b_2 = [0,1]^T\,, c_2 = \left[\frac{2}{3},2\right]^T\,;\\
A_3 &= [0,1;-2,1], b_3 = [0,1]^T, c_3 = [0.5,-1]^T;\\
A_4 &= [0,1,0;0,0,1;0.25,1,-0.25]\,, b_4 = [0,0,1]^T\,, c_4 = [0,0.25,1]^T\,.
\end{aligned}
$$

Their transfer functions $G_i(z) = k_{p_i} Z_i(z)/P_i(z),\ i = 1,2,3,4$, are computed as $P_1(z) = (z+1)(z-2)(z+0.5), Z_1(z) = z(z+0.5); P_2(z) = (z+1)(z-1), Z_2(z) = z+1/3; P_3(z) = (z+1)(z-2), Z_3(z) = -(z-0.5); P_4(z) = (z+0.25)(z+1)(z-1), Z_4(z) = z(z+0.25)$ with high-frequency gain parameters as $k_{p_1} = 1,\ k_{p_2} = 2,\ k_{p_3} = -1,\ k_{p_4} = 1$. The matrices of leader agent 0 are $A_0 = [0.6,0.3;-0.2,0.5], b_0 = [1,0.5]^T, c_0 = [1,0]^T$ and the external input $u_0(k)$ is set as $u_0(k) = 0.5\sin(0.05k) + 0.4\cos(0.12k)$. Moreover, the communication graph $\mathcal{G}$ is illustrated by Fig. 1.

Parameter setting. From the simulation model, we calculate the nominal parameters by the matching equation (3.2) as $\theta_{11}^* = [-1,-2.5,-1]^T,\ \theta_{12}^* = 1,\ \theta_{21}^* = [-1,-1/3]^T,\ \theta_{22}^* = 0.5,\ \theta_{31}^* = [-2,0]^T, \theta_{32}^* = -1, \theta_{41}^* = [-0.25,-1,0]^T, \theta_{42}^* = 1$. Similarly, we determine $\alpha_{01}^* = 0.6,\ \alpha_{02}^* = 1,\ \alpha_{11}^* = [1,2.5,1]^T,\ \alpha_{12}^* = 1,\ \alpha_{21}^* = [2,2/3]^T,\ \alpha_{22}^* = 2,\ \alpha_{31}^* = [-2,1]^T,\ \alpha_{32}^* = -1,\ \alpha_{41}^* = [2,2,0]^T,\ \alpha_{42}^* = 1$. Then, the ideal controller parameters $\theta_i^*,\ i = 1,2,3,4$, can be determined as

$$
\begin{aligned}
\theta_1^* &= [-1,-2.5,-1,0.6,0.3,1]^T\,, \theta_2^* = [-1,-1/3,0.3,0.15,0.5]^T\,,\\
\theta_3^* &= [-2,-0.5,-1,-2.5,-1,-2,-2/3,-1,-2]^T\,,\\
\theta_4^* &= [-0.25,-1,0,2,2/3,2,-0.5,2,-1]^T\,.
\end{aligned}
$$

We set the initial parameter estimate of θ_i^* as $\theta_i(0) = 70\%\theta_i^*,\ i = 1,2,3,4$, and $\rho_1(0) = 2,\ \rho_2(0) = 0.5,\ \rho_3(0) = -0.1,\ \rho_4(0) = 0.2$. The initial states of agents are chosen as $x_0(0) = 1,\ x_1(0) = [1,0,-1]^T,\ x_2(0) = [-0.5,0.5]^T,\ x_3(0) = [0,-0.2]^T,\ x_4(0) = [0.1,0,0]^T$. Moreover, the adaptive gains are selected as $\Gamma_1 = 0.9I,\ \Gamma_2 = 0.9I,\ \Gamma_3 = 1.2I,\ \Gamma_4 = 0.9I$ and $\gamma_1 = 0.8,\ \gamma_2 = 0.8,\ \gamma_3 = 1.5,\ \gamma_4 = 0.6$. From the values of high-frequency gain parameters k_{p_i}, we get $\mathrm{sign}\{k_{p_1}\} = 1,\ \mathrm{sign}\{k_{p_2}\} = 1,\ \mathrm{sign}\{k_{p_3}\} = -1,\ \mathrm{sign}\{k_{p_4}\} = 1$. With above parameter setting, it follows from (3.9) and (3.17) that the distributed MRAC law and the adaptive law can be specifically determined.

Simulation results. Simulation results are presented in Figs. 2-5. Figs. 2 and 3 show the agents' outputs $y_i(k),\ i = 0,1,2,3,4$, and the leader-following tracking errors $e_{i0}(k),\ i = 1,2,3,4$, respectively, which verify the asymptotic tracking performance under the distributed MRAC law (3.9). Fig. 4 depicts the control inputs $u_i(k),\ i = 1,2,3,4$, confirming boundedness of the closed-loop signals, while Fig. 5 illustrates the parameter estimates $\rho_i(k)$, which are consistent with Lemma 3.8. Overall, these results demonstrate the effectiveness of the proposed distributed adaptive control scheme with the distributed MRAC law (3.9) and the parameter update law (3.17).

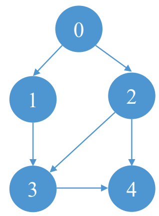

Fig. 1: Communication graph of MAS.

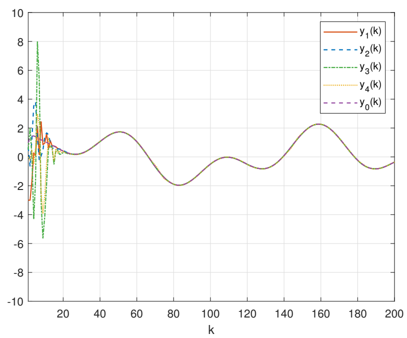

Fig. 2: System outputs $y_i(k)$ (control law (3.9)).

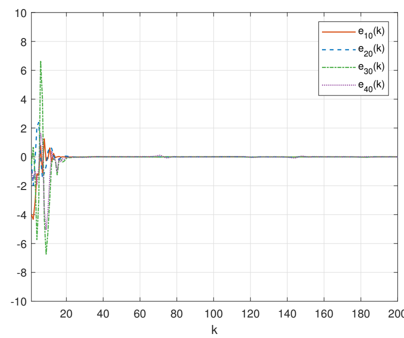

Fig. 3: Leader-following tracking errors $e_{i0}(k)$ (control law (3.9)).

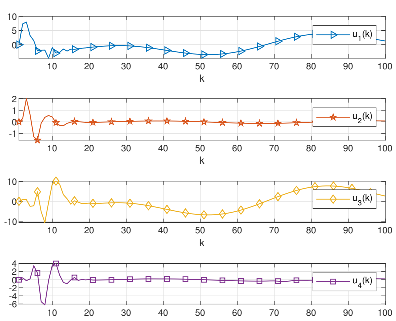

Fig. 4: Control inputs $u_i(k)$ (control law (3.9)).

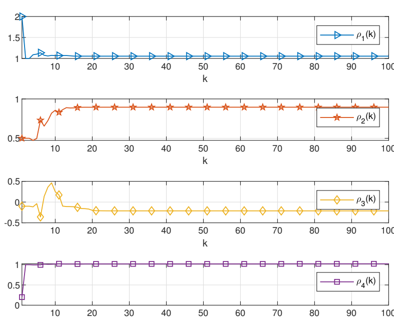

Fig. 5: Parameter estimates $\rho_i(k)$ (control law (3.9)).

4.2. Simulation for modified distributed MRAC. This subsection demonstrates the control performance under the modified distributed MRAC law (3.18) and the parameter update law (3.29).

Parameter setting. Still considering the same simulation model as above subsection, we set initial values for the modified distributed MRAC law (3.18) and parameter update law (3.29). Based on nominal values of θ_i^* provided in the preceding subsection, we compute the corresponding parameters β_i^* as

$$\beta_1^* = [-1, -2.5, -1, 0.6, 0.3, 1]^T, \beta_2^* = [-2, -2/3, 0.6, 0.3, 1]^T,$$
$$\beta_3^* = [2, 0.5, 1, 5/2, 1, 2, 2/3, 1, 2]^T, \beta_4^* = [-0.25, -1, 0, 2, 2/3, 2, -0.5, 2, -1]^T,$$

$\rho_1^* = 1, \rho_2^* = -1, \rho_3^* = 2, \rho_4^* = 1$ and $\mu_1^* = 1, \mu_2^* = -1, \mu_3^* = 1/2, \mu_4^* = 1$. Further, we obtain $\lambda_i^* = [\mu_i^*, \theta_i^{*T}, \beta_i^{*T}, \rho_i^*]^T$. We set initial estimates of parameters λ_i^* as $\lambda_i(0) = 70\%\lambda_i^*,\ i = 1, 2, 3, 4$. The initial states of agents are selected as $x_0(0) = 1,\ x_1(0) = [-1, 0, 1]^T,\ x_2(0) = [-0.5, 0]^T,\ x_3(0) = [0, 0.2]^T,\ x_4(0) = [-1, 0, 0]^T$. Moreover, the adaptive gains are selected as $\Upsilon_1 = 0.9I,\ \Upsilon_2 = 0.9I,\ \Upsilon_3 = 0.6I,\ \Upsilon_4 = 0.9I$. For the singularity-free design, we set the parameters $\underline{\kappa}_i = 1,\ i = 1, 2, 3, 4$. With above parameter setting, it follows from (3.18) and (3.29) that the modeified distributed MRAC law and the adaptive law can be specifically determined.

Simulation results. Simulation results are presented in Figs. 6-9. Figs. 6 and 7 show the agents' outputs $y_i(k),\ i = 0, 1, 2, 3, 4$, and the leader-following tracking errors $e_{i0}(k),\ i = 1, 2, 3, 4$, respectively, which verify the asymptotic tracking performance under the modified MRAC law (3.18). Fig. 8 depicts the control inputs $u_i(k),\ i = 1, 2, 3, 4$, while Fig. 9 illustrates the parameter estimates $\rho_i(k)$. These results confirm that the parameter estimates satisfy Lemma 3.17 and that all closed-loop signals remain bounded. Overall, Figs. 6–9 demonstrate the effectiveness of the proposed modified distributed MRAC scheme under the relaxed conditions (A1)-(A4).

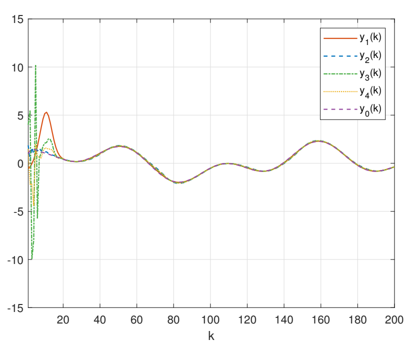

Fig. 6: System outputs $y_i(k)$ (control law (3.18)).

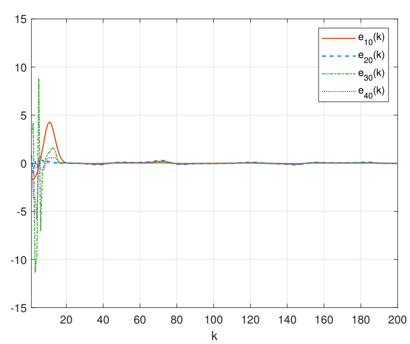

Fig. 7: Leader-following tracking errors $e_{i0}(k)$ (control law (3.18)).

Remark 4.1. In the simulation study, numerical example is employed to validate the effectiveness of the proposed distributed MRAC framework. Nevertheless, the developed method is formulated in a general state-space setting and is not restricted to numerical examples only. Specifically, the proposed approach is applicable to a class of practical systems with similar dynamic structures. For instance, the translational dynamics of unmanned aerial vehicles and the longitudinal dynamics of ground vehicles can often be approximated by second-order systems with uncertain parameters, which can be transformed into the form considered in this paper. However, such systems are more commonly investigated under formation control settings, where additional structural constraints and coordination objectives are imposed. Therefore, extending the proposed control scheme to unmanned aerial vehicle formations or vehicle platoons

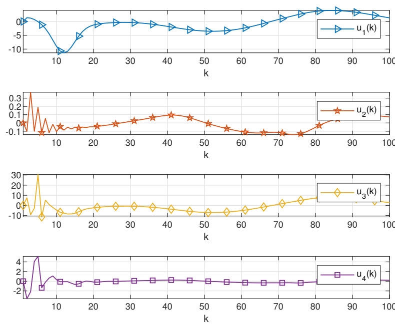

Fig. 8: Control inputs $u_i(k)$ (control law (3.18)).

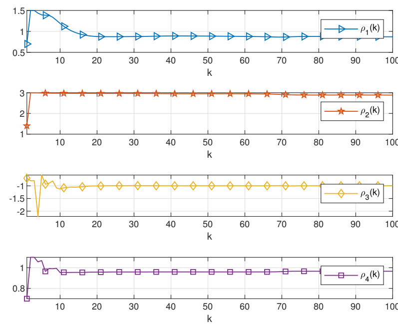

Fig. 9: Parameter estimates $\rho_i(k)$ (control law (3.18)).

may require modifications of the control protocol to incorporate formation patterns and system-specific characteristics. This will be considered as part of future work.

5. Concluding remarks. This work presents a systematic study of the distributed MRAC problem for a general class of uncertain discrete-time MASs. A distributed discrete-time MRAC framework is proposed. First, a nominal distributed MRC scheme with a unified controller structure is developed under known system parameters, where a distributed matching equation and a parameterized reconstruction of neighboring information are formulated. Then, an adaptive extension is constructed by incorporating parameter update laws to handle uncertainties in both leader and follower dynamics. Furthermore, a modified distributed MRAC scheme is proposed to eliminate the need for prior knowledge of control gains without relying on Nussbaum gain technique. The proposed schemes guarantee closed-loop stability and asymptotic leader-following output tracking. Future work will extend the proposed framework to more general settings with external disturbances or unmodeled dynamics to further investigate its robustness. In addition, extending the approach to multi-input multi-output agents remains a challenging problem for future research.

Appendix.

Proof of Theorem 3.9. First, we prove the control performance of follower agent 1. Under Assumption (A4), the agent 1 is only connected with the leader agent 0. Since the leader system (2.2) is stable and with bounded reference input $u_0(k)$, it yields that the state vector $x_0(k)$ and output signal $y_0(k)$ are bounded. Noting the form of the regressor $\omega_i(k)$ in (3.12), we get $||\omega_1(k)|| \le c \max_{k\le t} ||x_1(k)|| + c$ for the agent 1, where $c > 0$ is a generic constant bound. Then, since each follower agent is detectable and satisfies minimum phase property under Assumptions (A1) and (A2), with the boundedness of $y_0(k)$,we further obtain

$$||\omega_1(k)|| \le c \max_{t\le k} ||y_1(t)|| + c \max_{t\le k} ||u_1(t)|| + c \le c \max_{t\le k} ||e_1(t)|| + c. \tag{A.2}$$

Given $\bar{\omega}_i(k)$ in (3.12), it follows that

$$||\bar{\omega}_1(k)|| \le c \max_{t\le k-n^*} ||e_1(t)|| + c \le c \max_{t\le k} ||e_1(t)|| + c. \tag{A.3}$$

Then, from the definition of $\xi_i(k)$ in (3.12) and bounded property of $\theta_i(k)$ in Lemma 3.8, we obtain $||\xi_1(k)|| \le c\max_{t\le k}||e_1(t)||+c$, which combines (A.3) and (3.16) gives

$$||m_1(k)|| \le 1+||\bar{\omega}_1(k)||+||\xi_1(k)|| \le c\max_{t\le k}||e_1(t)||+c. \tag{A.4}$$

Moreover, from (3.14), we have

$$e_1(k) = \frac{\epsilon_1(k)}{m_1(k)}m_1(k) - \rho_1(k)(\theta_1^T(k) - \theta_1^T(k-n^*))\bar{\omega}_1(k). \tag{A.5}$$

By the property $\theta_i(k+1)-\theta_i(k) \in L^2 \cap L^\infty$ in Lemma 3.8, we derive that $\theta_1^T(k) - \theta_1^T(k-n^*) \in L^2 \cap L^\infty$. Noting that $\frac{\epsilon_i(k)}{m_i(k)} \in L^2 \cap L^\infty$ in Lemma 3.8, from the equations (A.3), (A.4) and (A.5), it yields $||e_1(k)|| \le \tau(k)\max_{t\le k}||e_1(t)||+c$, where $\tau(k)>0$ denotes a generic $L^2\cap L^\infty$ function satisfying $\lim_{k\to\infty}\tau(k)=0$. This implies $e_1(k) \in L^\infty$. Then, we obtain $y_1(k) \in L^\infty$ because the signal $y_0(k)$ is bounded. From the system model (2.4), we have $k_{p_1}z^{n^*}Z_1(z)[u_1](k) = P_1(z)[y_1](k+n^*)$, which implies that the input signal $u_1(k) \in L^\infty$ with $y_1(k) \in L^\infty$ and the stability of $z^{n^*}Z_1(z)$ under Assumption (A2). Then, we get the state variable $x_1(k)$ of agent 1 is also bounded with $y_1(k),\ u_1(k) \in L^\infty$ and the detectability of the system model (2.1) under Assumption (A1). Hence, all system signals related to the agent 1 are bounded. From Lemma 3.8, we have $\frac{\epsilon_1(k)}{m_1(k)} \in L^2\cap L^\infty$ and $\theta_1(k)-\theta_1(k-n^*) \in L^2\cap L^\infty$. Combining the equation (A.5), we obtain $e_1(k) \in L^2 \cap L^\infty$. This implies the local output tracking error $e_1(k)$ meets $\lim_{k\to\infty}e_1(k)=0$.

Next, we consider the follower agent 2. Under Assumption (A4), agent 2 may have three possible connectivity cases: (i) connected only to agent 0; (ii) connected only to agent 1; or (iii) connected to both agents 0 and 1. In all cases, due to the signal boundedness of agents 0 and 1, the regressor $\omega_2(k)$ satisfies $||\omega_2(k)|| \le c\max_{t\le k}||x_2(t)||+c$. Because the agent 2 is also a minimum phase system and detectable under Assumptions (A1) and (A2), we derive that $||\omega_2(k)|| \le c\max_{t\le k}||e_2(t)||+c$ with $y_0(k),\ y_1(k) \in L^\infty$. Then, it yields $||\bar{\omega}_2(k)|| \le c\max_{t\le k}||e_2(t)||+c$ from the definition of $\bar{\omega}_i(k)$. Further, we obtain that $||m_2(k)|| \le c\max_{t\le k}||e_2(t)||+c$. Therefore, with the properties of parameter estimates in Lemma 3.8, we have

$$\begin{aligned} ||e_2(k)|| &= \left|\left|\frac{\epsilon_2(k)}{m_2(k)}m_2(k) - \rho_2(k)(\theta_2(k)-\theta_2(t-n^*))\bar{\omega}_2(k)\right|\right| \\ &\le \left|\left|\tfrac{\epsilon_2(k)}{m_2(k)}\right|\right|\,||m_2(k)|| + ||\rho_2(k)(\theta_2(k)-\theta_2(t-n^*))||||\bar{\omega}_2(k)|| \\ &\le \tau(k)\max_{t\le k}||e_2(t)||+c, \end{aligned}$$

which indicates the local output tracking error $e_2(k)$ is bounded. Following the similar analysis as agent 1, it yields closed-loop signals involved in the agent 2 are bounded and $\lim_{k\to\infty}e_2(k)=0$.

In a similar fashion as above, we recursively prove closed-loop signals of all agents remain bounded and the local output tracking error $e_i(k)$ meets $\lim_{k\to\infty}e_i(k)=0,\ i=1,...,N$. Then, combining Lemma 3.4 gives $\lim_{k\to\infty}e_{i0}(k)=0,\ i=1,...,N$, which implies asymptotical leader-following tracking control objective is achieved with the distributed MRAC law (3.9) and the parameter update law (3.17). This completes the proof. □

Proof of Theorem 3.18. Similar to Theorem 3.9's proof procedure, we first consider the control performance of follower agent 1. According to Lemma 3.15 and

Lemma 3.17, $\kappa_i(k)$ is bounded and $\mu_i(k) + \kappa_i(k)$ is nonzero. With the signal boundedness of leader agent and the minimum phase property of follower agent, it follows from (3.25) that $||\psi_1(k)|| \le c \max_{t\le k+n^*} ||e_1(t)|| + c \max_{t\le k} ||\omega_1(t)|| + c$. Combing the equation (A.2), we obtain $||\psi_1(k)|| \le c \max_{t\le k+n^*} ||e_1(t)|| + c$, which together with $\bar{\psi}_i(k)$'s definition yields

$$||\bar{\psi}_1(k)|| \le c \max_{t\le k} ||e_1(t)|| + c. \tag{A.6}$$

Noting the equation (A.6), it follows from (3.29) that

$$||\bar{m}_1(k)|| \le 1 + ||\bar{\psi}_1(k)|| \le c \max_{t\le k} ||e_1(t)|| + c. \tag{A.7}$$

Based on the estimation error $\epsilon_i(k)$ in (3.27), it yields

$$\begin{aligned} e_1(k) &= \epsilon_1(k) - \lambda_1^T(k)\bar{\psi}_1(k) + \frac{1}{z^{n^*}}\left[\lambda_1^T \psi_1\right](k) \\ &= \frac{\epsilon_1(k)}{\bar{m}_1(k)}\bar{m}_1(k) - (\lambda_1^T(k) - \lambda_1^T(k-n^*))\bar{\psi}_1(k). \end{aligned} \tag{A.8}$$

According to the conclusion of Lemma 3.17, we have $\frac{\epsilon_1(k)}{\bar{m}_1(k)} \in L^2 \cap L^\infty$ and $\lambda_1^T(k) - \lambda_1^T(k-n^*) \in L^2 \cap L^\infty$. Then, combing the equations (A.6), (A.7) and (A.8), we obtain $||e_1(k)|| \le \left|\left|\frac{\epsilon_1(k)}{\bar{m}_1(k)}\right|\right| ||\bar{m}_1(k)|| + ||\lambda_1(k) - \lambda_1(k-n^*)|| ||\bar{\psi}_1(k)|| \le \tau(k) \max_{t\le k} ||e_1(t)|| + c$, which implies the local output tracking error $e_1(k)$ is bounded. Then, it yields $y_1(k) \in L^\infty$ via the bounedness of $y_0(k)$. From the system model (2.4), we have $k_{p_1} z^{n^*} Z_1(z)[u_1](k) = P_1(z)[y_1](k+n^*)$, which implies $u_1(k)$ is bounded with $y_1(k) \in L^\infty$ and the stability of $z^{n^*} Z_1(z)$ under Assumption (A2). Hence, the state vector $x_1(k)$ is also bounded with $y_1(k), u_1(k) \in L^\infty$ and Assumption (A1). Further, we derive that all involved system signals of agent 1 are bounded. Lemma 3.17 shows that $\frac{\epsilon_1(k)}{\bar{m}_1(k)} \in L^2 \cap L^\infty$ and $\lambda_1(k) - \lambda_1(k-n^*) \in L^2 \cap L^\infty$, which combines the equation (A.8) indicating that $e_1(k) \in L^2 \cap L^\infty$. This implies $\lim_{k\to\infty} e_1(k) = 0$.

Now we consider the follower agent 2. Under Assumption (A4), the control input of agent 2 is determined by exclusively the leader agent 0, the follower agent 1, or both. With signal boundedness of agents 0 and 1, following a similar analysis procedure as above, we derive the signal bounedness of agent 2 and $\lim_{k\to\infty} e_2(k) = 0$.

Recursively, we derive all agents in the MAS are closed-loop stable and the local output tracking error satisfies $\lim_{k\to\infty} e_i(k) = 0,\ i = 1, ..., N$. Finally, combining the conclusion of Lemma 3.4, we obtain $\lim_{k\to\infty} e_{i0}(k) = 0,\ i = 1, ..., N$. Thus, under the distributed adaptive control law (3.18) and the parameter update law (3.29), we achieve the leader-following asymptotical tracking control objective. This completes the proof. □

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